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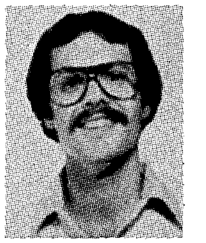

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The Generalized Lagrange Formulation for Nonlinear *RLC* Networks

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Abstract—**Based on the concept of generalized Euler–Lagrange equations, this paper develops a Lagrange formulation of *RLC* networks of considerably broad scope. It is shown that the generalized Lagrange equations along with a set of compatibility constraint equations represents a set of governing differential equations of order equal to the order of complexity of the network. In this method the generalized coordinates include capacitor charges and inductor fluxes and the generalized velocities are comprised of an independent set of capacitor voltages and inductor currents. The generalized Hamilton equations are also developed and the connection with the Brayton–Moser equations is established.**

I. INTRODUCTION

A CENTRAL issue in formulating Lagrange's equations for electrical networks, as in other types of physical systems, is the selection of generalized coordinates and velocities. The natural choice of the earliest Lagrange formulations of network equations was capacitor charge or inductor flux for coordinates and their formal derivatives for generalized velocities. Indeed, such a selection is used in most textbooks [1], [2] dealing with the subject and is legitimately referred to as the classical choice. The problem, of course, is to find a set of capacitor charges and/or inductor fluxes which satisfy the circuit topological constraints (admissability conditions) and along with these constraints completely specify the network.

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The usual procedure [1, for example] is to use either flux variables or charge variables but not both. In the former case the procedure is to identify a set of independent node voltages which are defined as the generalized velocities and their integrals (fluxes) are then the generalized coordinates. For the case of charge variables, the procedure is to identify a set of independent loop currents which are again defined to be generalized velocities and their integrals (charges) are then the generalized coordinates.

Although the method outlined above appears to be systematic and straightforward, the extent of its applicability is not at all clear. As a matter of fact, the procedure carries with it inherent limitations with regard to the type of components and topologies that can be treated. This is readily evident upon inspection of the worked examples in any standard text although the essential nature of the problem is never discussed. MacFarlane [3] took a major step towards clarifying the difficulty. He showed that if it were possible to choose a tree consisting entirely of inductors, then the inductor fluxes form a set of generalized coordinates in the spirit of the above procedure. Alternatively, if it is possible to choose a chord composed entirely of capacitors, then the capacitor charges form a set of generalized coordinates.

The possibility of relaxing the severe limitations of the above methods by the use of a mixed set of coordinates, i.e., both charges and fluxes, was considered by Chua and McPherson [4]. This pioneering work departed radically from conventional thinking. Their choice of coordinates was inductor charge and capacitor flux with respective velocities of inductor current and capacitor voltage. Chua and McPherson's work and subsequent extensions by Milić and Nov k [5] greately enlarged the range of applicability of Lagrangian methods to electrical networks. However, several questions of a fundamental nature are raised. Of central importance is the specification of initial conditions. The specification of capacitor charge and inductor flux is quite natural, but specification of capacitor flux or inductor charge is not. This point has been raised by Szatkowski [6]. Thus the question remains, is it necessary to abandon the classical choice of coordinates and, if so, why? In addition, for certain networks the Lagrangian of [4], [5] includes unspecified constant parameters related to the initial conditions. Once again it is necessary to ask whether this undesirable property is actually necessary and, if so, what is its meaning? These and related questions have not been previously addressed and form the motivation for the studies presented herein.

The development of the generalized Euler–Lagrange equations by Noble [7] allows the authors to view Lagrange's equations for electric circuits from a new perspective. It is shown that it is possible to return to the classical choice of capacitor charges and inductor fluxes as generalized coordinates. However, the generalized velocities are not simply the derivatives of the generalized coordinates. Rather, they are linear combinations of the coordinate derivatives and correspond specifically to a set of physical variables composed of capacitor voltages and inductor currents. The procedure described herein eliminates the aforementioned difficulties of Chua and McPherson's formulation.

The possible applicability of the generalized Lagrange formulation to electrical networks was suggested by Noble and Sewell [8] and Jones, Holding, and Evans [9].

In Sections II, III, and IV, certain assumptions are imposed for clarity which do not affect the general theory in any way. The networks contain only inductors and capacitors and we do not admit excess elements. *LC* networks illustrate the theory without the added complexities of converter elements. It is also assumed that all elements are bijective and in fact linear. This assumption is valuable in distinguishing two separate and important issues that can otherwise become confused. Previous authors attempt to treat the most general case from the onset and miss many insights into the nature of linear *LC* network problems. The extension to general nonlinear elements and the inclusion of resistors and independent sources is accomplished in Section V.

In Section VI we provide an example and in Section VII we establish the connection between the generalized Lagrange and Brayton–Moser equations. Section VIII deals with the generalized Hamilton equations and in Section IX the necessary modifications are made to incorporate excess elements and controlled sources.

Finally, we point out that generalized Euler–Lagrange equations of [7], [8] arise in the context of the theory of complementary and dual variational principles. We do not develop the variational aspect of the problem in this paper although we consider it to be an important and interesting issue. Indeed, as pointed out by one reviewer, our treatment of nonconservative elements through the addition of generalized forces (Section V) bypasses any link to variational principles, just as in the case of the classical Lagrange formulation.

II. The Generalized Lagrange Formulation

Given a dynamic network $\mathfrak{N}$ consisting of time invariant, linear capacitors, and inductors, choose a normal tree τ and let $\mathfrak{L}$ be its cotree. Recall that a normal tree is a tree containing a maximum number of capacitors and a minimum number of inductors [10]. We begin by restricting the discussion to networks without excess elements. That is, capacitor-only loops and inductor-only cutsets are not admitted. This condition will be relaxed in Section IX. If no excess elements are allowed then a normal tree contains no inductors. We further subdivide τ and $\mathfrak{L}$ into τ_1, τ_2 and $\mathfrak{L}_1, \mathfrak{L}_2$, respectively, and impose the following condition:

Hypothesis 1: elements in $\mathfrak{L}_2$ do not form fundamental loops with elements in τ_1.

Our perspective is to view the capacitor charges in τ_1, q_{c1}, and the inductor fluxes in $\mathfrak{L}_2, \phi_{L2}$, as generalized coordinates and the capacitor voltages in τ_2, v_{c2}, and the inductor currents in $\mathfrak{L}_1, i_{L1}$, as generalized velocities. In what follows we will establish the conditions under which such a point of view is appropriate.

What distinguishes this approach, of course, is our definition of generalized velocities, the implications of which will be discussed at length below. Central to the development of these ideas is the relationship between the generalized velocities and the derivatives of the generalized coordinates which we shall refer to as the *coordinate velocities*. We might also remark at this juncture that the important consequence of Hypothesis 1 is that the coordinate velocities $\dot{q}_{c1}$ are related only to the generalized velocities i_{L1} and not to any of the currents i_{L2}, and the coordinate velocities $\dot{\phi}_{L2}$ are related only to the generalized velocities v_{c2} and not to any of the voltages v_{c1}. Therefore, the relationship between our generalized coordinates and velocities assumes a particularly simple structure which will become evident later in this section when it is defined.

The tree and chord elements are related by the network dynamical transformation matrix [11] D as follows:

$$i_t = D i_c \tag{1}$$

$$v_c = -D' v_t \tag{2}$$

where

$$D = \begin{bmatrix} D_{ss} & D_{sn} & D_{se} \\ D_{ns} & D_{nn} & 0 \\ D_{es} & 0 & 0 \end{bmatrix} \tag{3}$$

i_t, v_t are the tree currents and voltages, respectively, and i_c, v_c are the chord currents and voltages, respectively. Due

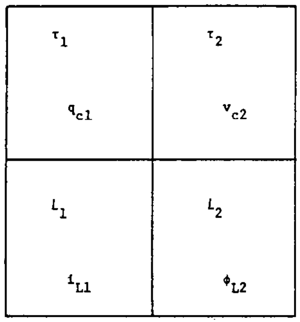

Fig. 1. Generalized coordinates and velocities.

to our assumption of no excess elements, D_{se} and D_{es} are null matrices. Also, D_{sn}, D_{ns}, and D_{nn} are null matrices since there are no converter elements. Therefore, (1) and (2) become

$$i_t = D_{ss} i_c \tag{4}$$

$$v_c = -D'_{ss} v_t. \tag{5}$$

Equation (4) can be subdivided to accommodate the different classes in Fig. 1, i.e.,

$$\begin{bmatrix} i_{c1} \\ i_{c2} \end{bmatrix} = \begin{bmatrix} D_{ss11} & 0 \\ D_{ss21} & D_{ss22} \end{bmatrix} \begin{bmatrix} i_{L1} \\ i_{L2} \end{bmatrix}. \tag{6}$$

Note that a zero appears in the top right corner of the D_{ss} matrix. This is due to the restriction that the inductor elements in $\mathfrak{L}_2$ when placed into the tree do not form loops which contain capacitor elements in τ_1. Similarly (5) can be written

$$\begin{bmatrix} v_{L1} \\ v_{L2} \end{bmatrix} = -\begin{bmatrix} D'_{ss11} & D'_{ss21} \\ 0 & D'_{ss22} \end{bmatrix} \begin{bmatrix} v_{c1} \\ v_{c2} \end{bmatrix}. \tag{7}$$

It is convenient to regroup (6) and (7) as follows. The upper part of (6) and the lower part of (7) provide a relationship between the generalized coordinates and velocities:

$$\frac{d}{dt}\begin{bmatrix} q_{c1} \\ \phi_{L2} \end{bmatrix} = A \begin{bmatrix} i_{L1} \\ v_{c2} \end{bmatrix} \tag{8}$$

where

$$A = \begin{bmatrix} D_{ss11} & 0 \\ 0 & -D'_{ss22} \end{bmatrix}.$$

The remaining equations are

$$v_{L1} = -D'_{ss11} v_{c1} - D'_{ss21} v_{c2} \tag{9a}$$

$$i_{c2} = D_{ss21} i_{L1} + D_{ss22} i_{L2}. \tag{9b}$$

Equation (8) relates the coordinate velocities and the generalized velocities. We shall refer to (8) as the *velocity transformation relation* and to the matrix A as the *velocity transformation matrix*. Moreover, (8) is to be used to uniquely establish the generalized velocities as a function of the coordinate velocities. This is trivially accomplished when A has an inverse, but when that is not the case we are led to several subtle and intriguing results which are delayed until Section IV.

Since the relationship between the generalized coordinates and velocities is not a classical one, it is reasonable to assume that the classical Lagrange equations may not be appropriate. This leads to the use of the so-called generalized Lagrange equations developed in Noble [7] and later appeared in Noble and Sewell [8]. The generalized Lagrange equations can be written

$$T\frac{\partial L}{\partial (T^*Q)'} + \frac{\partial L}{\partial Q'} = 0 \tag{10}$$

$$T^*Q = V \tag{11}$$

where

- Q generalized coordinate column vector,
- V generalized velocity column vector,
- L Lagrangian,
- T linear operator,
- T^* formal adjoint of T.

Equations (10) and (11) can be interpreted as the classical Lagrange equations when $T = -I(d/dt)$ which implies that $T^* = I(d/dt)$ [8]. The Lagrangian L is then the standard Lagrangian.

One point concerning notation is worth mention. We shall always define the Lagrangian L as a function of the generalized coordinates and generalized velocities, i.e., $L = L(V, Q)$. However, when L is used in (10) it is to be understood that T^*Q replaces V, that is, $L = L(T^*Q, Q)$. It is occasionally useful to write (10) in the form

$$T\frac{\partial L}{\partial V'} + \frac{\partial L}{\partial Q'} = 0 \tag{10a}$$

in which case it is intended that L be expressed in terms of V and Q. Equations (10) provide the differential governing equations. The interpretation (10a) will allow us to associate these equations directly with the loop and node equations (9).

Our major task is to define T and L appropriately to arrive at the correct equations of motion. In subsequent sections we shall show that this can indeed be accomplished and will prescribe a procedure for doing so. Furthermore, it is clear that (11) must represent the unique specification of generalized velocities in terms of coordinate velocities to be established from the velocity transformation relation, (8). In the following section we consider the simplest case where A has an inverse. The general case is considered in Section IV.

III. Invertible Velocity Transformation Matrix

In this section, it is assumed that the linear operator A has an inverse. It is true that A has an inverse if and only if D_{ss11} and D'_{ss22} have inverses. This implies that τ_1 and $\mathfrak{L}_1$ have the same number of elements and also that τ_2 and $\mathfrak{L}_2$ have the same number of elements. These are substantial restrictions on the allowable network configurations. However, this special case is a convenient vehicle for introducing several important ideas. We consider now the choice of the operator T and the definition of the Lagrangian.

The velocity transformation equation (8) can be written

$$\frac{d}{dt}Q=AV \tag{12}$$

where

$V=\begin{bmatrix} i_{L1} \\ v_{c2} \end{bmatrix}$, the generalized velocity vector

$Q=\begin{bmatrix} q_{c1} \\ \phi_{L2} \end{bmatrix}$, the generalized coordinate vector.

Since A^{-1} exists, (12) can be expressed

$$A^{-1}\frac{d}{dt}Q=V. \tag{13}$$

Upon comparison with (11) we are motivated to define the operator

$$T^*=A^{-1}\frac{d}{dt} \tag{14a}$$

and, therefore,

$$T=-(A^{-1})'\frac{d}{dt}. \tag{14b}$$

The generalized Lagrange equation (10) now becomes

$$-(A^{-1})'\frac{d}{dt}\left[\frac{\partial L}{\partial\left(A^{-1}\frac{d}{dt}Q\right)'}\right]+\frac{\partial L}{\partial Q'}=0. \tag{15}$$

Multiplying (15) by A' and replacing (13) into (15) yields

$$-\frac{d}{dt}\left[\frac{\partial L}{\partial V'}\right]+A'\frac{\partial L}{\partial Q'}=0. \tag{16}$$

Replacing V and Q by their components and dividing (16) into node and loop equations, the following equations result:

$$-\frac{d}{dt}\left[\frac{\partial L}{\partial i'_{L1}}\right]+D'_{ss11}\frac{\partial L}{\partial q'_{c1}}=0 \tag{17a}$$

$$-\frac{d}{dt}\left[\frac{\partial L}{\partial v'_{c2}}\right]-D_{ss22}\frac{\partial L}{\partial \phi'_{L2}}=0. \tag{17b}$$

We now consider a class of quadratic Lagrangians defined as follows:

$$L=\frac{1}{2}V'\begin{bmatrix}\tilde{L} & 0\\ 0 & \tilde{C}\end{bmatrix}V-\frac{1}{2}Q'\begin{bmatrix}\tilde{S} & 0\\ 0 & \tilde{\Gamma}\end{bmatrix}Q+Q'GV \tag{18}$$

where

- $\tilde{L}$ submatrix of the inductance matrix L,
- $\tilde{S}$ submatrix of the elastance matrix S,
- $\tilde{\Gamma}$ submatrix of the inverse inductance matrix Γ,
- $\tilde{C}$ submatrix of the capacitance matrix C,
- G cross product matrix.

The Lagrangian of (18) is a standard one except for the cross product term, $Q'GV$. The cross product term is necessary when $D_{ss21}\neq 0$. As can be seen in MacFarlane [3] and Jones and Evans [12], their Lagrangian formulation leads only to loop equations or node equations. Therefore, no cross product term appears. A cross product term does appear in Chua and McPherson [4] and Milic and Novak [5] but there is no discussion as to its necessity or how it was chosen. In the following we will show how to define the cross product term and will show that it is not unique.

Our purpose is to demonstrate that the formulation described above does indeed lead to the network equations of motion with an appropriate choice of G. Rewriting (18) and inserting the components of V and Q, results in

$$L=\frac{1}{2}i'_{L1}\tilde{L}i_{L1}+\frac{1}{2}v'_{c2}\tilde{C}v_{c2}-\frac{1}{2}\phi'_{L2}\tilde{\Gamma}\phi_{L2}-\frac{1}{2}q'_{c1}\tilde{S}q_{c1}+\begin{bmatrix}q_{c1}\\ \phi_{L2}\end{bmatrix}'\begin{bmatrix}G_{11} & G_{12}\\ G_{21} & G_{22}\end{bmatrix}\begin{bmatrix}i_{L1}\\ v_{c2}\end{bmatrix}. \tag{19}$$

Evaluating (17a) using the Lagrangian of the form of (19) yields

$$-\tilde{L}\frac{di_{L1}}{dt}-G'_{21}\frac{d\phi_{L2}}{dt}-D'_{ss11}\tilde{S}q_{c1}+D'_{ss11}G_{12}v_{c2}+(D'_{ss11}G_{11}-G'_{11}D_{ss11})i_{L1}=0. \tag{20}$$

However, (20) is a voltage loop equation and cannot contain any currents. Therefore, the last term in (20) must be zero for all i_{L1} at any time, t. Thus results in the first restriction for the G matrix, i.e.,

$$D'_{ss11}G_{11}-G'_{11}D_{ss11}=0. \tag{21}$$

Equation (20) becomes

$$-\tilde{L}\frac{di_{L1}}{dt}-G'_{21}\frac{d\phi_{L2}}{dt}-D'_{ss11}\tilde{S}q_{c1}+D'_{ss11}G_{12}v_{c2}=0. \tag{22}$$

From (8),

$$\frac{d}{dt}\phi_{L2}=-D'_{ss22}v_{c2}. \tag{23}$$

Replacing (23) into (22) and collecting terms yields

$$-\tilde{L}\frac{di_{L1}}{dt}-D'_{ss11}\tilde{S}q_{c1}+(D'_{ss11}G_{12}+G'_{21}D'_{ss22})v_{c2}=0. \tag{24}$$

Substituting the appropriate constitutive relationships into (24), it is easily seen that (24) is the topological voltage loop equation (9a) provided

$$-D'_{ss21}=(D'_{ss11}G_{12}+G'_{ss22}D'_{ss22}). \tag{25}$$

Equation (25) is the second restriction for the matrix G.

Evaluating (17b) using the Lagrangian of (19) yields the following equation:

$$-\tilde{C}\frac{dv_{c2}}{dt}-G'_{12}\frac{dq_{c1}}{dt}+D_{ss22}\tilde{\Gamma}\phi_{L2}-D_{ss22}G_{21}i_{L1}+(G'_{22}D'_{ss22}-D_{ss22}G_{22})v_{c2}=0. \tag{26}$$

However, (26) is a current node equation and cannot contain any voltages. Therefore, the last term in (26) must be zero for all v_{c2} at any time, t. This results in the third restriction for the G matrix, i.e.,

$$G'_{22}D'_{ss22}-D_{ss22}G_{22}=0. \tag{27}$$

Equation (26) becomes

$$-\tilde{C}\frac{dv_{c2}}{dt}-G'_{12}\frac{dq_{c1}}{dt}+D_{ss22}\tilde{\Gamma}\phi_{L2}-D_{ss22}G_{21}i_{l1}=0. \tag{28}$$

From (8),

$$\frac{dq_{c1}}{dt}=D_{ss11}i_{L1}. \tag{29}$$

Replacing (29) into (28) and collecting terms yields

$$-\tilde{C}\frac{dv_{c2}}{dt}+D_{ss22}\tilde{\Gamma}\phi_{L2}-(D_{ss22}G_{21}+G'_{12}D_{ss11})i_{L1}=0. \tag{30}$$

Substituting the appropriate constitutive relationships into (30), it is easily seen that this equation is the topological current node equation (9b) provided

$$-D_{ss21}=(D_{ss22}G_{21}+G'_{21}D_{ss11}). \tag{31}$$

So far we have the following. The choice of T and L yields the correct loop and node equations provided the four restrictions, (21), (25), (27), and (31), are observed. These can be arranged into the following matrix equation:

$$\begin{bmatrix} D'_{ss11} & 0 \\ 0 & -D_{ss22}\end{bmatrix}\begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22}\end{bmatrix}-\begin{bmatrix} G'_{11} & G'_{21} \\ G'_{12} & G'_{22}\end{bmatrix}\cdot\begin{bmatrix} D_{ss11} & 0 \\ 0 & -D'_{ss22}\end{bmatrix}=\begin{bmatrix} 0 & -D'_{ss21} \\ D_{ss21} & 0\end{bmatrix}. \tag{32}$$

Equation (32) can be written as follows:

$$A'G-(A'G)'=M \tag{33}$$

where M is a skew symmetric matrix defined in (32). Equation (33) states a well-known fact that a square matrix minus its transpose yields a skew-symmetric matrix. However, another fact is that a square matrix plus its transpose yields a symmetric matrix. Therefore,

$$(A'G)+(A'G)'=N \tag{34}$$

where N is an arbitrary symmetric matrix.

The solution for G is now apparent. Adding (33) and (34) yields

$$A'G=\tfrac{1}{2}(M+N). \tag{35}$$

Therefore, since A has an inverse, then

$$G=\tfrac{1}{2}(A')^{-1}(M+N). \tag{36}$$

It is now obvious that G is not unique since N is arbitrary. Equation (36) can produce Chua and McPherson's [4] and Milić and Novak's [5] cross product term with the proper choice of N.

For this section, the choice of N and consequently G is as follows:

$$N=\begin{bmatrix} 0 & D'_{ss21} \\ D_{ss21} & 0\end{bmatrix} \tag{37}$$

$$G=\begin{bmatrix} 0 & 0 \\ -D_{ss22}^{-1}D_{ss21} & 0\end{bmatrix}. \tag{38}$$

Therefore, if A has an inverse the only cross product term is $-\phi'_{L2}D_{ss22}^{-1}D_{ss21}i_{L1}$. The Lagrangian of (19) is then

$$L=\tfrac{1}{2}i'_{L1}\tilde{L}i_{L1}+\tfrac{1}{2}v'_{c2}\tilde{C}v_{c2}-\tfrac{1}{2}\phi'_{L2}\tilde{\Gamma}\phi_{L2}-\tfrac{1}{2}q'_{c1}\tilde{S}q_{c1}-\phi'_{L2}D_{ss22}^{-1}D_{ss21}i_{L1}. \tag{39}$$

The equations of motion can be obtained by replacing the definition of G, (38), into (30) and (24). This produces the following equations:

$$-\tilde{C}\frac{dv_{c2}}{dt}+D_{ss22}\tilde{\Gamma}\phi_{L2}+D_{ss21}i_{L1}=0 \tag{40a}$$

and

$$-\tilde{L}\frac{di_{L1}}{dt}-D'_{ss11}\tilde{S}q_{c1}-D'_{ss21}v_{c2}=0. \tag{40b}$$

In order to solve (40) it is necessary to obtain equations of motion in terms of the generalized coordinates and their derivatives. Therefore, by replacing (13) into (40), the following equations are obtained:

$$\tilde{C}(D'_{ss22})^{-1}\ddot{\phi}_{L2}+D_{ss22}\tilde{\Gamma}\phi_{L2}+D_{ss21}D_{ss11}^{-1}\dot{q}_{c1}=0 \tag{41a}$$

and

$$-\tilde{L}D_{ss11}^{-1}\ddot{q}_{c1}-D'_{ss11}\tilde{S}q_{c1}+D'_{ss21}(D'_{ss22})^{-1}\dot{\phi}_{L2}=0. \tag{41b}$$

Notice that the initial values necessary to solve these equations are $q_{c1}(0)$, $\phi_{L2}(0)$, $v_{c2}(0)$, and $i_{L1}(0)$ since $\dot{q}_{L1}$ and $\dot{\phi}_{L2}$ can be computed from v_{c2} and i_{L1} using the velocity transformation relation, (13). All these quantities are physically meaningful.

IV. The General Velocity Transformation Matrix

The purpose of this section is to extend the procedure described above to the general case where the velocity transformation matrix A does not have an inverse. Our objective is to use the velocity transformation relation, (12), to establish a unique specification of the generalized velocities in terms of the coordinate velocities. There are two difficulties. The first is that there may not exist any solutions of (12) for V. We shall see that solutions for V exist only if certain "compatibility" constraints are imposed on the coordinate velocities. The second problem is that if a solution to (12) exists it may not be unique. In general it will be necessary to extend the coordinate vector in order to assure a unique solution. In such a case we can associate with A a pseudo-inverse A^+ which has the property

$$AA^+A=A. \tag{42}$$

Furthermore, let

$$A=LR \tag{43}$$

be a rank factorization of A where L is a matrix $m\times k$ and R is a matrix $k\times n$. L and R can always be found for any $m\times n$ matrix A of rank k [13]. Since L and R are of full rank they possess, respectively, left and right inverses L^l

and R^r. A pseudo-inverse of A is then

$$A^+ = R^r L^l. \tag{44}$$

We further define the $n \times (n-k)$ and $(n-k) \times n$ matrices Δ and Φ by a rank factorization of $\{I - R^r R\}$:

$$\{I - R^r R\} = \Delta \Phi. \tag{45}$$

Similarly, define the $m \times (m-k)$ and $(m-k) \times m$ matrices Γ, Σ by a rank factorization of $\{I - LL^l\}$:

$$\{I - LL^l\} = \Gamma \Sigma. \tag{46}$$

Note the following properties of these matrices:

$$\begin{bmatrix} R \\ \Phi \end{bmatrix} [R^r \quad \Delta] = I \tag{47}$$

$$\begin{bmatrix} L^l \\ \Sigma \end{bmatrix} [L \quad \Gamma] = I. \tag{48}$$

It is now possible to summarize the solution properties of (12). A solution of (12) exists if and only if

$$\Sigma \dot{Q} = 0. \tag{49}$$

If (49) holds, then all solution of (12) take the form

$$V = R^r L^l \dot{Q} + \Delta \dot{w} \tag{50}$$

where $\dot{w}$ is an arbitrary $(n-k)$ vector.

It is convenient to collect (49) and (50) into the form

$$\begin{bmatrix} V \\ 0 \end{bmatrix} = \begin{bmatrix} R^r L^l & \Delta \\ \Sigma & 0 \end{bmatrix} \begin{bmatrix} \dot{Q} \\ \dot{w} \end{bmatrix} \tag{51}$$

and note its inverse

$$\begin{bmatrix} \dot{Q} \\ \dot{w} \end{bmatrix} = \begin{bmatrix} LR & \Gamma \\ \Phi & 0 \end{bmatrix} \begin{bmatrix} V \\ 0 \end{bmatrix}. \tag{52}$$

For convenience, define the matrices:

$$\mathscr{A} = \left[\tilde{A} \,\vdots\, \Lambda \right] = \begin{bmatrix} LR & \Gamma \\ \Phi & 0 \end{bmatrix} \tag{53}$$

and

$$\mathscr{A}^{-1} = \begin{bmatrix} \tilde{A}^l \\ \psi \end{bmatrix} = \begin{bmatrix} R^r L^l & \Delta \\ \Sigma & 0 \end{bmatrix} \tag{54}$$

and also the *extended coordinate vector*, $\tilde{Q}$,

$$\tilde{Q} = \begin{bmatrix} Q \\ w \end{bmatrix}. \tag{55}$$

The variables w will be referred to as *quasi-coordinates.* Equation (52) can now be written

$$\dot{\tilde{Q}} = \tilde{A} V. \tag{56}$$

We will refer to (56) as the *extended velocity relation.* Equation (51) can be written

$$\tilde{A}^l \dot{\tilde{Q}} = V \tag{57}$$

$$\psi \dot{\tilde{Q}} = 0. \tag{58}$$

Equations (56) and (57) define the relationship between the generalized velocities and coordinate velocities in both directions. The *compatibility constraints*, (58), must hold for V to exist. Comparing (57) with (11) motivates the definition

$$T^* \triangleq \tilde{A}^l \frac{d}{dt} \tag{59a}$$

and, therefore,

$$T = -(\tilde{A}^l)' \frac{d}{dt}. \tag{59b}$$

As in the previous case in which the velocity transformation matrix was invertible, the velocity transformation relation, (45), leads to the definition of generalized velocities in terms of the operator T^*, (57) and (59a). However, in this case we are also led to an extension of the number of generalized coordinates (55) and to a set of constraint or compatibility equations, (58). It should be noted that if the velocity transformation matrix has a left inverse then no additional coordinates are required. On the other hand, if it has a right inverse then there are no compatibility constraints.

The compatibility constraints are of differential form and since they are obviously integrable they are by definition holonomic. However, if (58) were integrated (as is common practice when dealing with holonomic constraints) the result would introduce precisely the same number of arbitrary constants as the number of generalized coordinates we would hope to eliminate. Consequently there would not be a reduction in the number of degrees of freedom. Alternatively, the method of Lagrange multipliers can be used to incorporate these differential constraints.

Lagrange's equations must be altered to accommodate the use of Lagrange multipliers [14]. We shall consider the generalized (10) modified as follows:

$$T \frac{\partial L}{\partial (T^* \tilde{Q})'} + \frac{\partial L}{\partial \tilde{Q}'} = \psi' \lambda \tag{60}$$

where ψ is defined in (57), λ is a column vector of Lagrange multipliers, and the equations are written in terms of the extended coordinate vector $\tilde{Q}$. Using the definitions of T, T^* given in (59) leads to

$$-(\tilde{A}^l)' \frac{d}{dt} \left(\frac{\partial L}{\partial (\tilde{A}^l \dot{\tilde{Q}})'} \right) + \frac{\partial L}{\partial \tilde{Q}'} = \psi' \lambda. \tag{61}$$

Making use of (57) and premultiplying (61) by $\mathscr{A}$ defined in (56) yields

$$-(\tilde{A}^l \tilde{A})' \frac{d}{dt} \left(\frac{\partial L}{\partial V'} \right) + \tilde{A}' \frac{\partial L}{\partial \tilde{Q}'} = (\psi \tilde{A})' \lambda \tag{62a}$$

$$-(\tilde{A}^l \Lambda)' \frac{d}{dt} \left(\frac{\partial L}{\partial V'} \right) + \Lambda' \frac{\partial L}{\partial \tilde{Q}'} = (\psi \Lambda)' \lambda. \tag{62b}$$

Using the properties of Λ (following (55)), (62) reduces to

$$-\frac{d}{dt} \left(\frac{\partial L}{\partial V'} \right) + \tilde{A}' \frac{\partial L}{\partial \tilde{Q}'} = 0 \tag{63a}$$

$$\Lambda' \frac{\partial L}{\partial \tilde{Q}'} = \Lambda. \tag{63b}$$

Equation (63b) explicitly provides the solution for the

Lagrange multipliers λ, whereas (63a) does not contain λ. Moreover, (63a) can be divided into two parts upon use of the partitionings of V, $\tilde{Q}$, and $\tilde{A}$

$$-\frac{d}{dt}\left(\frac{\partial L}{\partial i'_{L1}}\right)+D'_{ss11}\frac{\partial L}{\partial q'_{c1}}+\Phi'_1\frac{\partial L}{\partial w'}=0 \quad (64a)$$

$$-\frac{d}{dt}\left(\frac{\partial L}{\partial v'_{c2}}\right)-D_{ss22}\frac{\partial L}{\partial \phi'_{c2}}+\Phi'_2\frac{\partial L}{\partial w'}=0. \quad (64b)$$

We have yet to define the Lagrangian and to show that (64) along with the compatibility equation (58) do provide the governing equations for the network. Again we consider a class of quadratic Lagrangians, now of the form

$$L=\tfrac{1}{2}i'_{L1}\tilde{L}i_{L1}+\tfrac{1}{2}v'_{c2}\tilde{C}v_{c2}-\tfrac{1}{2}\phi'_{L2}\tilde{\Gamma}\phi_{L2}-\tfrac{1}{2}q'_{c1}\tilde{S}q_{c1}+\begin{bmatrix} q_{c1}\\ \phi_{L2}\\ w\end{bmatrix}\begin{bmatrix} G_{11} & G_{12}\\ G_{21} & G_{22}\\ G_{31} & G_{32}\end{bmatrix}\begin{bmatrix} i_{L1}\\ v_{c2}\end{bmatrix}. \quad (65)$$

Notice that the new coordinates, w, appear only in the cross product term and, as before, the first four terms in (65) correspond to the "kinetic" energy minus the "potential" energy and depend only on the original generalized coordinates and generalized velocities.

As in Section III the determination of the G matrix can be obtained by evaluating (64) using the Lagrangian of (65). This development follows that of Section III closely with similar results except that (35) is extended to incorporate the w coordinates. Therefore, the equation for the cross product term is

$$\tilde{A}'G=\tfrac{1}{2}(M+N) \quad (66)$$

where M and N are the same matrices as in Section III (equations (32), (37)). Since $\tilde{A}$ has a left inverse, (66) can be easily solved for G as follows:

$$G=\tfrac{1}{2}(\tilde{A}^{l})'(M+N). \quad (67)$$

Evaluating (67) using (37) for N, G is obtained:

$$G=\begin{bmatrix} 0 & 0\\ (-D'_{ss22})^{+'}D_{ss21} & 0\\ \Delta'_2 D_{ss21} & 0\end{bmatrix}. \quad (68)$$

Equation (65) becomes

$$L=\tfrac{1}{2}i'_{L1}\tilde{L}i_{L1}+\tfrac{1}{2}v'_{c2}\tilde{C}v_{c2}-\tfrac{1}{2}\phi'_{L2}\tilde{T}\phi'_{L2}-\tfrac{1}{2}q'_{c1}\tilde{S}q_{c1}-\phi'_{L2}(D'_{ss22})^{+'}D_{ss21}i_{L1}+w'_1\Delta'_2D_{ss21}i_{L1}. \quad (69)$$

The Lagrangian (69) can be employed in (64) to obtain

$$-\tilde{L}\frac{di_{L1}}{dt}-D'_{ss21}\left(-D'^{+}_{ss22}\dot{\phi}_{L2}+\Delta_2\dot{w}\right)-D'_{ss11}\tilde{S}q_{c1}+\Phi'_1\Delta'_2D_{ss21}i_{L1}=0 \quad (70a)$$

$$-\tilde{C}\frac{dv_{c2}}{dt}+D_{ss22}\left(\tilde{\Gamma}\phi_{L2}+(D'_{ss22})^{+'}D_{ss21}i_{L1}\right)+\Phi'_2\Delta'_2D_{ss21}i_{L1}=0. \quad (70b)$$

In order to proceed it is useful to note certain identities.

Expanding (52), it can be seen that

$$\begin{matrix} s\updownarrow\\ t\updownarrow\end{matrix}\overset{\quad s \qquad\qquad t}{\begin{bmatrix} D^{+}_{ss11}D_{ss11} & 0\\ 0 & (D'_{ss22})^{+}D'_{ss22}\end{bmatrix}}+\begin{bmatrix}\Delta_1\phi_1 & \Delta_1\phi_2\\ \Delta_2\phi_1 & \Delta_2\phi_2\end{bmatrix}=\begin{bmatrix} I_s & 0\\ 0 & I_t\end{bmatrix} \quad (71)$$

where I_s and I_t are identity matrices of dimension $s\times s$ and $t\times t$, respectively. Therefore,

$$D^{+}_{ss}D_{ss11}+\Delta_1\Phi_1=I_s \quad (72a)$$

$$\Delta_1\Phi_2=0_{t\times s} \quad (72b)$$

$$\Delta_2\Phi_1=0_{t\times s} \quad (72c)$$

$$(D'_{ss22})^{+}D'_{ss22}+\Delta_2\Phi_2=I_t\cdot \quad (72d)$$

From the standard velocity relation, (54), we have

$$\dot{q}_{c1}=D_{ss11}i_{L1} \quad (73a)$$

$$\dot{q}_{L2}=-D'_{ss22}v_{c2} \quad (73b)$$

$$\dot{w}=\Phi_1 i_{L1}+\Phi_2 v_{c2}. \quad (73c)$$

Now, using (72c), (73b), (73c), and (72d) in (70a) yields

$$-\tilde{L}\frac{di_{L1}}{dt}-D'_{ss11}\tilde{S}q_{c1}-D'_{ss21}v_{c2}=0. \quad (74a)$$

Using (72d) in (70b) yields

$$-\tilde{C}\frac{dv_{c2}}{dt}+D_{ss22}\tilde{\Gamma}\phi_{L2}+D_{ss21}i_{L1}=0. \quad (74b)$$

Equations (74a) and (74b) are recognized as sets of loop and node equations, respectively. Compare (74a) with (9a) and (74b) with (9b).

It remains to be shown that the mixed set of loop and node equations, (74a) and (74b), along with the compatibility equations (58) form a complete set of network equations. To see this we rewrite (74a) and (74b) in terms of the generalized coordinates and their derivatives by using the definition of generalized velocities, (57), in the form

$$i_{L1}=D^{+}_{ss11}\dot{q}_{c1}+\Delta_1\dot{w} \quad (75a)$$

$$v_{c2}=-(D'_{ss22})^{+}\dot{\phi}_{L2}+\Delta_2\dot{w}. \quad (75b)$$

Using these relation in (74a) and (74b) yields

$$-\tilde{L}D^{+}_{ss11}\ddot{q}_{c1}-\tilde{L}\Delta_1\ddot{w}_1-D'_{ss11}\tilde{S}q_{c1}+D'_{ss11}\tilde{S}q_{c1}+D'_{ss21}(D'_{ss22})^{+}\dot{\phi}_{L2}-D'_{ss21}\Delta_2\dot{w}=0 \quad (76a)$$

$$\tilde{C}(D'_{ss22})^{+}\ddot{\phi}_{L2}-\tilde{C}\Delta_2\ddot{w}_1+D_{ss22}\tilde{\Gamma}\phi_{L2}+D_{ss21}D^{+}_{ss11}\dot{q}_{c1}+D_{ss21}\Delta_1\dot{w}=0. \quad (76b)$$

Equations (76) can be reduced in order by defining

$$z=\dot{w}=\phi_1 i_{L1}+\phi_2 v_{c2}.$$

Therefore, (76) can be written

$$-\tilde{L}D^{+}_{ss11}\ddot{q}_{c1}-\tilde{L}\Delta_1\dot{z}-D'_{ss11}\tilde{S}q_{c1}+D'_{ss21}(D'_{ss22})^{+}\dot{\phi}_{L2}-D'_{ss21}\Delta_2 z=0 \quad (77a)$$

and

$$\tilde{C}(D'_{ss22})^{+}\ddot{\phi}_{L2}-\tilde{C}\Delta_2\dot{z}+D_{ss22}\tilde{\Gamma}\phi_{L2}+D_{ss21}D^{+}_{ss11}\ddot{q}_{c1}+D_{ss21}\Delta_1 z=0. \quad (77b)$$

For completeness (58) can be rewritten as

$$\Psi_1\dot{q}_{c1}+\Psi_2\dot{\phi}_{L2}+\Psi_3 z=0. \quad (77c)$$

We now summarize several important facts. First, (77a) and (77b) comprise n equations and (77c) represents an additional $m-k$ equations for a total of $n+m-k$. The vectors q_{c1}, ϕ_{L2} comprise m variables and z constitutes an additional $n-k$ for a total of $n+m-k$ variables.

Equations (77) require $2m+n-k$ initial conditions. Specifically, these are $q_{c1}(0)$, $\dot{q}_{c1}(0)$, $\phi_{L2}(0)$, $\dot{\phi}_{L2}(0)$, and $z(0)$. Note that from (73) $i_{L1}(0)$ and $v_{c2}(0)$ provide $\dot{q}_{c1}(0)$, $\dot{\phi}_{L2}(0)$, and $z(0)$. Consequently, the initial values $q_{c1}(0)$, $\phi_{L2}(0)$, $i_{L1}(0)$, and $v_{c2}(0)$ completely specify the initial conditions required for (77).

The solution of (77) provides q_{c1}, $\dot{q}_{c1}$, ϕ_{L2}, $\dot{\phi}_{c2}$, and z for all t. Equations (75) allow i_{L1} and v_{c2} to be computed from $\dot{q}_{c1}$, $\dot{\phi}_{L2}$ and z. Thus the set of differential equations (77) specifies all tree branch voltages (or capacitor charges) and chord currents (or inductor fluxes), i.e., completely solves the network.

Thus, the formulation prescribed above is complete in the sense that specification of all capacitor charges or voltages and inductor fluxes or currents specifies the initial conditions required to solve (77) which in turn provides a complete solution of the network.

V. Nonlinear and Converter Elements

The previous discussion was restricted to linear, LC circuits. These limitations will now be removed. First, our formulation will be extended to nonlinear LC elements and subsequently extended to include resistors and source elements.

Nonlinear Elements

Since general nonlinear elements may not be bijective we place further restrictions on the element classes defined in Section II to account for the causality of the constitutive relationships:

Hypothesis 2: τ_1 does not contain any voltage controlled capacitors and τ_2 does not contain any charge controlled capacitors. Similarly, $\mathfrak{L}_1$ does not contain any flux controlled inductors and $\mathfrak{L}_2$ does not contain any current controlled inductors.

With these addition restrictions on the classification of network elements the previous discussions regarding system topology remain applicable. Specifically, the definition of the operators T^*, T, (59) remain unchanged. The Lagrangian, (65), however, is modified as follows. Following Cherry [17] define the capacitor co-energy, inductor co-energy, capacitor energy, and inductor energy, respectively:

$$T^*_{c\tau 2}(v_{c2})=\int q'_{c2}(v_{c2})\,dv_{c2} \quad (78a)$$

$$U^*_{L\mathfrak{L}1}(i_{L1})=\int \phi'_{L1}(i_{L1})\,di_{L1} \quad (78b)$$

$$T_{c\tau 1}(Q_{c1})=\int v'_{cL}(q_{c1})\,dq_{c1} \quad (78c)$$

$$U_{L\mathfrak{L}2}(\phi_{L2})=\int i'_{L2}(\phi_{L2})\,d\phi_{L2}. \quad (78d)$$

Now, the Lagrangian is

$$L(V,\tilde{Q})=W^*(V)-Z(Q)+\tilde{Q}'GV \quad (79a)$$

where

$$W^*(V)=T^*_{c\tau 2}(v_{c2})+U^*_{L\mathfrak{L}1}(i_{L1}) \quad (79b)$$

represents the total co-energy associated with the capacitors in τ_2 and inductors in $\mathfrak{L}_1$, and

$$Z(q)=T_{c\tau 1}(q_{c1})+U_{L\mathfrak{L}2}(\phi_{L2}) \quad (79c)$$

represents the total energy associated with the capacitors in τ_1 and inductors in $\mathfrak{L}_2$. Note that the cross product term in (79a) remains the same as that of the linear case, (65). This is to be anticipated since the cross product matrix G depends only on the topology and not on the constitutive relationships.

Using the Lagrangian, (79), and evaluating (64) leads to the following nonlinear counterparts to (77):

$$-\frac{\partial\phi_{L1}}{\partial i_{L1}}\left(D^{+}_{ss11}\ddot{q}_{c1}+\Delta_1\dot{z}\right)+D'_{ss21}\left[(D'_{ss22})^{+}\dot{\phi}_{L2}-\Delta_2 z\right]-D'_{ss11}v_{c1}(q_{c1})=0 \quad (80a)$$

$$\frac{\partial q_{c2}}{\partial v_{c2}}\left[(D'_{ss22})^{+}\ddot{\phi}_{L2}-\Delta_2\dot{z}\right]+D_{ss22}\left[D^{+}_{ss11}\dot{q}_{c1}+\Delta_1 z\right]+D_{ss22}i_{L2}(\phi_{L2})=0. \quad (80b)$$

Note that $\partial\phi_{L1}/\partial i_{L1}$ is a function of i_{L1} and can thus be expressed as a function of $\dot{q}_{c1}$ and z using (75a). Similarly $\partial q_{c2}/\partial v_{c2}$ is a function of v_{c2} and can be expressed as a function of $\dot{\phi}_{L2}$ and z using (75b).Finally, the compatibility constraints, (77c), remain unchanged and completes the set of network governing equations. The discussion following (77) applies here as well.

Converter Elements

The results of the previous section will now be extended to include independent current and voltage sources and nonlinear resistive elements. To accomplish this it is necessary to place additional restrictions on the classification of elements defined in Section II. We assume that following:

Hypothesis 3: All independent voltage sources belong to τ_1 and all independent current sources belong to $\mathfrak{L}_2$.

Hypothesis 4: All resistors are divided between τ_1 and $\mathfrak{L}_2$ such that all current-controlled resistors belong to τ_1 and all voltage-controlled resistors belong to $\mathfrak{L}_2$.

It is possible to approach the inclusion of nonconservative elements as contemplated here in either of two ways: through modification of the Lagrangian or through the use of so-called generalized forces. We shall take the latter course. In order to see what is required, the topological network equations (1) and (2) are written as follows, taking into account the topological restrictions specified in Hypotheses 1, 2, 3, 4:

$$\begin{bmatrix} i_{c1} \\ i_{c2} \\ i_{E1} \\ i_{R\tau 1} \end{bmatrix} = \begin{bmatrix} D_{ss11} & 0 & 0 & 0 \\ D_{ss21} & D_{ss22} & D_{sN1} & D_{sN2} \\ D_{Ns1} & 0 & 0 & 0 \\ D_{Ns2} & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} i_{L1} \\ i_{L2} \\ i_{J2} \\ i_{R\mathcal{L}2} \end{bmatrix} \tag{81a}$$

$$\begin{bmatrix} v_{L1} \\ v_{L2} \\ v_{J2} \\ v_{R\mathcal{L}2} \end{bmatrix} = -\begin{bmatrix} D'_{ss11} & D'_{ss21} & D'_{Ns1} & D'_{Ns2} \\ 0 & D'_{ss22} & 0 & 0 \\ 0 & D'_{sN1} & 0 & 0 \\ 0 & D'_{sN2} & 0 & 0 \end{bmatrix} \begin{bmatrix} v_{c1} \\ v_{c2} \\ v_{E1} \\ v_{r\tau 1} \end{bmatrix}. \tag{81b}$$

Note that the velocity transformation relation (8) is a subset of (81) and remains the basis for specification of the velocity transformation matrix A, and consequently for the operators T^*, T. There is no change in our choice of generalized coordinates nor velocities. Furthermore, the Lagrangian is not changed from (79).

In view of the earlier discussion it is to be anticipated that Lagrange's equations will produce the loop equation which specifies v_{L1} and the node equation which specifies i_{c2}, i.e.,

$$v_{L1} = -D'_{ss11}v_{c1} - D'_{ss21}v_{c2} - D'_{Ns1}V_{E1} - D'_{Ns2}v_{R\tau 1} \tag{82a}$$

$$i_{c2} = D_{ss21}i_{L1} + D_{ss22}i_{L2} + D_{sN1}i_{J2} + D_{sN2}i_{R\mathcal{L}2}. \tag{82b}$$

Notice that the last two terms of each equation are newly added (compared to (9)) and will not be generated by Lagrange's equation in their present form, (60). Consequently, we modify, (60) to include generalized forces, $F(V,t)$:

$$T\frac{\partial L}{\partial (T^*\dot{Q})'} + \frac{\partial L}{\partial \tilde{Q}'} = \Psi'\lambda + (\tilde{A}')'F. \tag{83}$$

Just as in the development of (63), premultiply (95) by $\mathcal{Q}$ as defined in (56) to obtain

$$-\frac{d}{dt}\left(\frac{\partial L}{\partial V'}\right) + \tilde{A}'\frac{\partial L}{\partial \tilde{Q}'} = F \tag{84a}$$

$$\Lambda'\frac{\partial L}{\partial \tilde{Q}'} = \lambda. \tag{84b}$$

Finally, note that the left side of (84a) generates the left side of (80) and, furthermore, these correspond to all but the last two terms of each of (82a) and (82b). These must be provided by F. Thus define

$$F(V,t) = \begin{bmatrix} F_1(i_{L1},t) \\ F_2(v_{c2},t) \end{bmatrix} = \begin{bmatrix} D'_{Ns1}v_{E1}(t) + D'_{Ns2}v_{R\tau 1}(D_{Ns2}i_{L1}) \\ -D_{sN1}i_{J2}(t) - D_{sN2}i_{R\mathcal{L}2}(-D'_{sN2}v_{c2}) \end{bmatrix}. \tag{85}$$

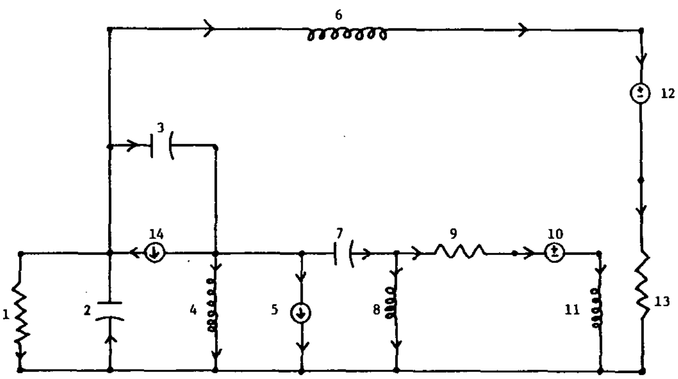

Fig. 2.

We simply remark that the equations of motion for the general case are composed of (80a) and (80b) with the right-hand sides replaced by F_1 and F_2, respectively, with (75a) and (75b) used to replace i_{L1} by $\dot{q}_{c1}$ and z and v_{c2} by ϕ_{L2} and z. In addition the unaltered compatibility constraints, (77c) complete the set. Since (73) remain valid, the initial values $q_{c1}(0)$, $\phi_{L2}(0)$, $i_{L1}(0)$, and $v_{c2}(0)$ completely specify the initial conditions required to solve the governing differential equations. Moreover, once those equations are solved, $i_{L1}(t)$ and $v_{c2}(t)$ are recovered from (75) and all remaining unknown tree currents and chord voltages can be determined from the topological equations (81). Constitutive relations provide any other variables.

VI. Example

The network Fig. 2, is taken from Chua and McPherson [4]. It is an example in their paper and serves as an example of the method proposed herein and as a comparison with their method. There is one change to the network. The controlled sources are replaced by independent sources. The constitutive relationships are as follows:

$$\begin{array}{lll} i_1 = (v_1+2)^3 & \phi_6 = i_6^3 & \phi_{11} = 4i_{11} \\ q_2 = 8v_2^3 & v_7 = (q_7+1)^5 & v_{12} = 3\text{ V} \\ q_3 = v_3^5 & \phi_8 = (i_8-2)^5 & v_{13} = \tfrac{1}{2}i_{13}^2 \\ i_4 = \phi_4^7 + 1 & v_9 = 3i_9^5 & i_{14} = 5\text{ A}. \\ i_5 = 2\text{ A} & v_{10} = 20\text{ V} & \end{array}$$

For the tree τ choose the set of branches $\{2,3,7,9,10,12,13\}$. Let $\tau_1 = \{7,9,10,12,13\}$ and $\tau_2 = \{2,3\}$. Then $\mathcal{L}_1 = \{6,8,11\}$ and $\mathcal{L}_2 = \{1,4,5,14\}$. With this choice, the generalized coordinates are q_7 and ϕ_4. The generalized velocities ae i_6, i_8, i_{11}, v_2, and v_3. The topological current relationships are

$$\begin{bmatrix} i_7 \\ i_2 \\ i_3 \\ i_{10} \\ i_{12} \\ i_9 \\ i_{13} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} i_6 \\ i_8 \\ i_{11} \\ i_4 i_5 \\ i_{14} \\ i_1 \end{bmatrix}. \tag{86}$$

From (86), the following are obtained:

$$D_{ss11}=[0\quad 1\quad 1]$$

$$D_{ss21}=\begin{bmatrix}1&1&1\\0&1&1\end{bmatrix}$$

$$D_{ss22}=\begin{bmatrix}1\\1\end{bmatrix}$$

$$D_{SN1}=\begin{bmatrix}1&0\\1&1\end{bmatrix}$$

$$D_{SN2}=\begin{bmatrix}1\\0\end{bmatrix}$$

$$D_{NS1}, D_{NS2}=\begin{bmatrix}0&0&1\\1&0&0\end{bmatrix}.$$

A is constructed as follows:

$$\dot{Q}=\begin{bmatrix}\dot{q}_7\\\dot{\phi}_4\end{bmatrix}=\begin{bmatrix}0&1&1&0&0\\0&0&0&-1&-1\end{bmatrix}\begin{bmatrix}i_6\\i_8\\i_{11}\\v_2\\v_3\end{bmatrix}=AV. \tag{87}$$

Note that A has a right inverse and

$$A^r=\begin{bmatrix}D^r_{ss11}&0\\0&(-D'_{ss22})^r\end{bmatrix}\begin{bmatrix}0&0\\1&0\\0&0\\0&0\\0&-1\end{bmatrix}.$$

Next, define

$$\begin{aligned}\Delta&=\text{independent columns }\{I-A^rA\}\\&=\begin{bmatrix}1&0&0\\0&-1&0\\0&0&0\\0&0&1\\0&0&-1\end{bmatrix}=\begin{bmatrix}\Delta_1\\\Delta_2\end{bmatrix}.\end{aligned}$$

Therefore,

$$(\tilde{A})^{-1}=\left[A^r\vdots\Delta\right]=\begin{bmatrix}0&0&1&0&0\\1&0&0&-1&0\\0&0&0&0&0\\0&0&0&0&1\\0&-1&0&0&-1\end{bmatrix}$$

and

$$\tilde{A}=\begin{bmatrix}A\\\phi_1\quad\phi_2\end{bmatrix}=\begin{bmatrix}0&1&1&0&0\\0&0&0&-1&-1\\1&0&0&0&0\\0&0&1&0&0\\0&0&0&1&0\end{bmatrix}.$$

The Lagrangian is

$$L=\int q_2\,dv_2+\int q_3\,dv_3+\int\phi_6\,di_6+\int\phi_8\,di_8+\int\phi_{11}\,di_{11}$$
$$-\phi_4(D'_{ss22})^{r'}D_{ss21}[i_6i_8i_{11}]'+[w_1w_2w_3]\,\Delta'_2D_{ss21}[i_6i_{11}]'. \tag{88}$$

Recall that

$$V=A^r\dot{Q}+\Delta\dot{w}$$

or

$$\begin{bmatrix}i_6\\i_8\\i_{11}\end{bmatrix}=\begin{bmatrix}0\\1\\0\end{bmatrix}\dot{q}_7+\begin{bmatrix}1&0&0\\0&-1&0\\0&0&0\end{bmatrix}\begin{bmatrix}\dot{w}_1\\\dot{w}_2\\\dot{w}_3\end{bmatrix} \tag{89}$$

and

$$\begin{bmatrix}v_2\\v_3\end{bmatrix}=\begin{bmatrix}0\\-1\end{bmatrix}\dot{\phi}_4+\begin{bmatrix}0&0&1\\0&0&-1\end{bmatrix}\begin{bmatrix}\dot{w}_1\\\dot{w}_2\\\dot{w}_3\end{bmatrix}. \tag{90}$$

Lagrange's equations are (with $z=\dot{w}$, $\dot{z}=\ddot{w}$)

$$7z_1^6\dot{z}_1-z_3-3=0.5z_1^2 \tag{91a.1}$$
$$5(\dot{q}_7-z_2-2)^4(q_7-\dot{z}_2)+\dot{\phi}_4-(q_7+1)^5=0 \tag{91a.2}$$
$$4\dot{z}_2+\dot{\phi}_4-(\ddot{q}_7+1)^5-20=3z_3^5 \tag{91a.3}$$
$$-24z_3\dot{z}_3+\dot{q}_7+z_1+(\phi_4^7+1)+2=(z_3-2)^3 \tag{91b.1}$$
$$-5(\dot{\phi}_4+q_3)^4(\ddot{\phi}_4+\dot{z}_3)+\dot{q}_7+(\phi_4^7+1)+7=0. \tag{91b.2}$$

Equations (91) can be solved for q_7, ϕ_4, z_1, z_2, and z_3 given the initial conditions $q_7(0)$, $\phi_4(0)$, $i_6(0)$, $i_8(0)$, $i_{11}(0)$, $v_2(0)$, and $v_3(0)$, (54). Once q_j, ϕ_4, z_1, z_2, and z_3 are found then i_6, i_8, i_{11}, v_2, and v_3 can be found as functions of time using (89) and (90).

Note the following differences with the method of Chua and McPherson [4]. Their Lagrangian contains initial conditions which this method does not. These initial conditions can be viewed as undefined parameters and are not functions of the generalized coordinates or velocities or time. The method of [4] results in five second-order equations which constitute a tenth-order system. Since the order of complexity is seven, all that is needed is a seventh-order system. Our method produces such a system. Chua and McPherson have two initial conditions in their tenth-order system which implies 5 "extra" degrees of freedom. They state that "additional constraint equations have not been applied." They are five in number bringing their system to seventh order in principle.

VII. The Brayton–Moser Equations

Again following Cherry we define the resistor and voltage source content $G_{R\tau 1}$ and $G_{E\tau 1}$, and resistor and current source co-content $G^*_{R\ell 2}$ and $G^*_{J\ell 2}$:

$$G_{R\tau 1}(i_{R1})=\int v'_{r1}(i_{R1})\,di_{R1} \tag{92a}$$

$$G_{E\tau 1}(i_{E1},t)=\int v'_{E1}(t)\,di_{E1} \tag{92b}$$

$$G^*{}_{R\ell 2}(v_{R2})=\int i'_{R2}(v_{R2})\,dv_{R2} \tag{92c}$$

$$G^*_{J\ell 2}(v_{J2},t)=\int i'_{J2}(t)\,dv_{J2}. \tag{92d}$$

The generalized forces $F_1(i_{L1},t)$ and $F_2(v_{c2},t)$ of (97) can be expressed

$$F_1(i_{L1},t)=\frac{\partial}{\partial i'_{L1}}\left[G_{E\tau 1}(D_{Ns1}i_{L1},t)+G_{R\tau 1}(D_{Ns2}i_{L1})\right]. \tag{93a}$$

$$F_2(v_{c2},t)=\frac{\partial}{\partial v'_{c2}}\left[G^*_{J2}(-D'_{sN1}v_{c2},t)+G^*_{R\mathcal{L}2}(-D'_{sN2}v_{c2})\right]. \tag{93b}$$

If all capacitors are voltage controlled and all inductors current controlled then it is possible to place all capacitors in τ_2 and all inductors in $\mathcal{L}_1$. In this degenerate case, all generalized coordinates are quasi-coordinate defined via the extension process (that is $\{\dot{Q}\}$ is vacuous) and we have

$$\tilde{A}=\Phi=I \quad \tilde{A}^l=\Delta=I \tag{94}$$

and the extended velocity relation becomes

$$\dot{\tilde{Q}}=w=V. \tag{95}$$

The Lagrangian assumes the form

$$L=W^*(V)+w'GV \tag{96}$$

and Lagrange's equations are

$$-\frac{d}{dt}\left(\frac{\partial L}{\partial V}\right)+\frac{\partial L}{\partial w}=F. \tag{97}$$

In view of (96) these can be expressed

$$\frac{\partial^2 W^*}{\partial V^2}\frac{dV}{dt}=(G'-G)V-F=MV-F. \tag{98}$$

Let $n_{L\mathcal{L}1}$ and $n_{c\tau 2}$ denote the number of inductors in $\mathcal{L}_1$ and capacitors in τ_2, respectively, and define the matrix

$$J=\operatorname{diag}\left(I_{n_{L\mathcal{L}1}},-I_{n_{c\tau 2}}\right) \tag{99}$$

and also the "mixed potential function:"

$$\begin{aligned}P(V,t)=P(i_{L1},v_{c2},t)&=v'_{c2}D_{ss21}i_{L1}-G^*_{J\mathcal{L}2}(-D'_{sN1}v^t_{c21})\\&-G^*_{R\mathcal{L}2}(-D'_{sN2}v_{c2})+G_{E\tau 1}(D_{Ns1}i_{L1},t)\\&+G_{R\tau 1}(D_{Ns2}i_{L1}).\end{aligned} \tag{100}$$

It is easily verified that (98) can be written

$$\left[\frac{\partial^2 W^*}{\partial V^2}\right]\frac{dV}{dt}=J\frac{\partial P}{\partial V'} \tag{101}$$

which are the Brayton–Moser equations, [15], [16]. In partioned form (101) reduces to

$$\left[\frac{\partial \phi_{L1}}{\partial i_{L1}}\right]\frac{di_{L1}}{dt}=\frac{\partial P}{\partial i_{L1}} \tag{102a}$$

$$\left[\frac{\partial q_{c2}}{\partial v_{c2}}\right]\frac{dv_{c2}}{dt}=-\frac{\partial P}{\partial v_{c2}}. \tag{102b}$$

This degenerate case has a special connection to previous work [4], [5]. It is the link between Chua and McPherson's [4] formulation and the formulation of this paper. Our degenerate case precisely matches the Chua and McPherson formulation if they were to choose that all capacitors and all inductors contribute a generalized coordinate. This selection is the only one in which their Lagrangian does not contain initial conditions.

VIII. The Generalized Hamilton Equations

In the usual way, we define the Hamilton, $H(P,\tilde{Q})$, via a Legendre transformation of the Lagrangian $L(V,\tilde{Q})$:

$$P\triangleq\frac{\partial L}{\partial V'} \tag{103}$$

$$H(P,\tilde{Q})\triangleq P'V-L(V,\tilde{Q}) \tag{104}$$

where it is necessary that the implicit relation (103) uniquely defines V in terms of P and it is understood that V is eliminated from (104) using (103). The transformation defined by (103) and (104) also possesses the properties:

$$V=\frac{\partial H}{\partial P'} \tag{105}$$

and

$$\frac{\partial H}{\partial \tilde{Q}}=-\frac{\partial L}{\partial \tilde{Q}}. \tag{106}$$

In view of (103) and (106), (83) can be written

$$TP=\frac{\partial H}{\partial \tilde{Q}'}+\psi'\lambda(\tilde{A}^l)'F. \tag{107a}$$

Equation (105) can be expressed

$$T^*\tilde{Q}=\frac{\partial H}{\partial P'}. \tag{107b}$$

Equations (107) are the generalized Hamilton equations and along with the constraint, (58), provide a complete description of the network. Using the definitions of T, T^*, (59), it is convenient to rewrite (107) and (58) in the form

$$-(\tilde{A}^l)'\dot{P}=\frac{\partial H}{\partial \tilde{Q}'}+\psi'\lambda+(\tilde{A}^l)'F \tag{108a}$$

$$\begin{bmatrix}\tilde{A}^l\\ \Psi\end{bmatrix}\dot{\tilde{Q}}=\begin{bmatrix}\dfrac{\partial H}{\partial P'}\\ 0\end{bmatrix}. \tag{108b}$$

Premultiplying (108b) by A as defined in (56) and (108a) by A' yields

$$-\dot{P}=\tilde{A}'\frac{\partial H}{\partial \tilde{Q}'}+F \tag{109a}$$

$$\dot{\tilde{Q}}=\tilde{A}\frac{\partial H}{\partial P'} \tag{109b}$$

$$\lambda=-\Lambda'\frac{\partial H}{\partial \tilde{Q}'}. \tag{109c}$$

Equations (109a) and (109b) constitute a complete description of the network and are a useful alternative form of the generalized Hamilton equations. Equation (109c) is identical to (63b) and explicitly provides the constraint forces in terms of P and $\tilde{Q}$.

The Hamiltonian can be explicitly obtained using the definition, (103), (104) and the Lagrangian, (79) as follows.

The momentum is

$$P=\frac{\partial L}{\partial V'}=\frac{\partial W^*}{\partial V'}+G'\tilde{Q} \tag{110}$$

and the Hamiltonian is

$$H(P,\tilde{Q})=\frac{\partial W^*}{\partial V}V+\tilde{Q}'GV-W^*(V)+Z(Q)-\tilde{Q}'GV$$
$$=W(P-G'\tilde{Q})+Z(Q). \tag{111}$$

Use has been made of the fact that

$$W\left(\frac{\partial W^*}{\partial V}\right)=\frac{\partial W^*}{\partial V}V-W^*(V). \tag{112}$$

In view of (111), it is clear that the Hamiltonian can be interpreted as the total circuit stored energy.

It is useful to compute

$$\dot{H}(P,\tilde{Q})=\frac{\partial H}{\partial P}\dot{P}+\frac{\partial H}{\partial \tilde{Q}}\dot{\tilde{Q}}=-V'F(V,t). \tag{113}$$

Equations (111) and (113) can be used to determine stability properties of network.

By a simple change of variables the quasi-coordinates, w, can be seen to be ignorable. As an alternative to P we can define the "momenta" variables π:

$$\pi \triangleq \frac{\partial W^*}{\partial V}=P-G'\tilde{Q} \tag{114}$$

in terms of which the Hamiltonian can be expressed

$$\tilde{H}(\pi,\tilde{Q})\triangleq H(P,\tilde{Q})|_{P=\pi+G'\tilde{Q}}=W(\pi)+Z(\tilde{Q}). \tag{115}$$

Note that (109a) and (109b) can be written in terms of π as

$$\dot{\pi}=(\tilde{A}'G-G'\tilde{A})\frac{\partial W}{\partial \pi'}-\tilde{A}'\frac{\partial Z}{\partial \tilde{Q}'}-F \tag{116a}$$

$$\dot{\tilde{Q}}=\tilde{A}\frac{\partial W}{\partial \pi'}. \tag{116b}$$

Straightforward computation shows that

$$\tilde{A}'G-G'\tilde{A}=M \tag{117}$$

where M is defined by (22) and (33). Moreover, since Z depends only on Q we can ignore the quasi-coordinates and rewrite (116) as

$$\dot{\pi}=M\frac{\partial W}{\partial \pi'}-A\frac{\partial Z}{\partial Q'}-F \tag{118a}$$

$$\dot{Q}=A\frac{\partial W}{\partial \pi}. \tag{118b}$$

Equations (118) provide a complete description of the circuit, however, the Hamiltonian structure has been lost.

IX. Excess Elements and Controlled Sources

In this section we extend the formulation to networks which contain excess elements and controlled sources. If the network contains a capacitor-only loop, then one of the capacitors must be placed in the chord. Similarly, one of the inductors in an inductor only cutset must be placed in the tree. When excess elements are present we require the following:

Hypothesis 5: All inductors in the tree are current controlled and belong to τ_1. All capacitors in the chord are voltage controlled an belong to $\mathfrak{L}_2$.

In order to deal with controlled sources we replace Hypothesis 3 by:

Hypothesis 3': All voltage sources belong to τ_1 with controlling current belonging to an element in $\tau_1\cup\mathfrak{L}_1$. All current sources belong to $\mathfrak{L}_2$ with controlling voltage belonging to an element in $\tau_2\cup\mathfrak{L}_2$.

The tree and chord variables are related as follows:

$$i_c=\begin{bmatrix} i_{c1}\\ i_{c2}\\ i_{E1}\\ i_{R\tau1}\\ i_{L\tau1}\end{bmatrix}=\begin{bmatrix} D_{ss11} & 0 & 0 & 0 & 0\\ D_{ss21} & D_{ss22} & D_{sN1} & D_{sN2} & D_{sE}\\ D_{Ns1} & 0 & 0 & 0 & 0\\ D_{Ns2} & 0 & 0 & 0 & 0\\ D_{Es} & 0 & 0 & 0 & 0\end{bmatrix}\cdot\begin{bmatrix} i_{L1}\\ i_{L2}\\ i_{J2}\\ i_{R\mathfrak{L}2}\\ i_{c\mathfrak{L}2}\end{bmatrix}=Di_t \tag{119a}$$

and

$$v_c=-D'v_t. \tag{119b}$$

The velocity transformation relation and matrix, (108), remain unchanged for networks which include excess elements and controlled sources. Therefore, the discussion of Section II remains valid. The excess elements do not contribute any coordinates or velocities to the formulation. However, they do contribute two co-energy terms to the Lagrangian. Define the co-energy functions for the excess elements

$$T_{cL2}(v_{c2})=\int q'_{c\mathfrak{L}2}(v_{c\mathfrak{L}_2})\,dv_{c\mathfrak{L}2}\Big|_{v_{c\mathfrak{L}2}=-D_{se}v_{c2}} \tag{120a}$$

$$U^*_{L\tau1}(i_{L1})=\int \phi'_{L\tau_1}(i_{L\tau_1})\,di_{L\tau_1}\Big|_{i_{L\tau_1}=D_{es}i_{L1}}. \tag{120b}$$

The Lagrangian is as before (eq. (79)) except that

$$W^*(V)=T^*_{c\tau2}(v_{c2})+U^*_{L\mathfrak{L}1}(i_{L1})+T^*_{c\mathfrak{L}_2}(v_{c2})+U^*_{L\tau1}(i_{L1}). \tag{121}$$

The fact that the sources are controlled has no effect on the velocity transformation matrix or the Lagrangian. These elements influence the equations only through the generalized forces $F(V,t)$. It is necessary to show that the controlled source voltages and currents depend only on V, i.e., i_{L1} and v_{c2}, and t. From (119) it is evident that all voltages belonging to elements in $\tau_2\cup\mathfrak{L}_2$ are directly related to v_{c2} and, similarly, all currents belonging to elements in $\tau_1\cup\mathfrak{L}_1$ are directly related to i_{L1}. Hence,

$$i_{J2}=i_{J2}(v_{c2},t) \tag{122a}$$

$$v_{E1}=v_{E1}(i_{L1},t) \tag{122b}$$

as was illustrated by Chua and McPherson [4]. With this modification (85) remains valid.

X. Summary and Conclusions

In this paper we have established a generalized Lagrange representation for a broad class of nonlinear *RLC* circuits with independent sources. It has been shown that the generalized Lagrange formulation leads to a system of ordinary differential equations in a mixed set of generalized coordinates including capacitor charges (q_{c1}) and inductor fluxes (ϕ_{L2}). Moreover, the equations were shown to be a subset of the network loop and node equations along with certain additional compatibility equations. Furthermore, the initial conditions required for solution of the differential equations were shown to be completely specified by knowledge of the initial charge or voltage of each capacitor and the initial flux or current of each inductor. Finally, we established that the solution of the differential equations did specify all network variables.

The generalized Lagrange equations employed here include Lagrange multipliers to handle differential constraints and generalized forces to handle the nonconservative effects. Including the compatibility constraints, the equations take the form

$$T\frac{\partial L}{\partial (T^*\tilde{Q})'}+\frac{\partial L}{\partial \tilde{Q}'}=\psi'\lambda+(\tilde{A}^{l})'F(T^*\tilde{Q},t)$$

$$\Psi\dot{\tilde{Q}}=0$$

or, equivalently,

$$-\frac{d}{dt}\left(\frac{\partial L}{\partial (T^*\tilde{Q})'}\right)+\tilde{A}'\frac{\partial L}{\partial \tilde{Q}'}=F(T^*\tilde{Q},t)$$

$$\Lambda'\frac{\partial L}{\partial \tilde{Q}'}=\lambda$$

$$\Psi\dot{\tilde{Q}}=0.$$

The generalized coordinates, $\tilde{Q}$, are composed of capacitor charges, q_{c1}, inductor fluxes, ϕ_{L2}, and additional variables, w, termed quasi-coordinates, whose derivatives are linear combinations of the generalized velocities. The generalized velocities, V, comprise capacitor voltages, v_{c2}, and inductor currents, i_{L1}. Furthermore, the relationship between the generalized velocities and coordinates is $V=T^*\tilde{Q}$, where the differential operator T^* has been defined in the text.

The introduction of quasi-coordinates are necessary when the generalized velocities are not uniquely defined by the coordinate velocities, $\dot{q}_{c1}$ and $\dot{\phi}_{L2}$. This could be simply a matter of choice in selecting a particular partitioning of network elements. Compatibility constraints arise if the network contains inductor loops or capacitor cutsets, and/or by virtue of the partitioning of elements.

It has been shown that the Lagrangian is composed of the usual energy storage terms plus a velocity-coordinate cross product term. Such cross product terms were noted to appear in earlier mixed formulations of network Lagrangians. The cross product term is shown to be nonunique and an explicit procedure for selecting it has been developed.

The link with the method of Chua and McPherson was shown to be through our "degenerate" case—a case which can arise only in special circumstances. Moreover, these circumstances represent the only situation in which the Lagrangian of Chua and McPherson does not contain arbitrary parameters related to initial conditions.

We have shown, in Section VII, that if the network element constitutive equations have the appropriate causality then the Brayton–Moser equations can be derived from the generalized Lagrange equations. In Section VIII we develop the generalized Hamilton equations in the form

$$TP=\frac{\partial H}{\partial \tilde{Q}'}+\psi'\lambda+(\tilde{A}^{l})'F$$

$$T^*\tilde{Q}=\frac{\partial H}{\partial P'}$$

$$\psi\dot{\tilde{Q}}=0$$

or, equivalently,

$$-\dot{P}=\tilde{A}'\frac{\partial H}{\partial \tilde{Q}'}+F$$

$$\dot{\tilde{Q}}=\tilde{A}\frac{\partial H}{\partial P}$$

$$\lambda=-\Lambda'\frac{\partial H}{\partial \tilde{Q}'}.$$

We have also shown that the quasi-coordinates are ignorable so that a minimal order differential system is obtained whose order is precisely the order of complexity of the network. In Section IX we extend the earlier results to networks with excess elements and controlled sources.

Again we should note that we have dealt only with the formulative aspects of the generalized Lagrange and Hamilton equations for nonlinear *RLC* networks and have not touched upon their association with variational principles.

Finally, we should point out that the restrictions imposed by Hypotheses 1, 2, 3, 4 are precisely those of [4], [5]. These constraints can become binding when there are nonbijective elements and/or purely converter multiports within the network. Consequently, there is motivation for future studies to relax these conditions.

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